# GENERATORS IN $l_1$ [1]

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In an earlier paper (see [4]) the authors investigated the following problem: Let $f(z) = \sum C_n z^n$ where $\sum |C_n| < \infty$ (i.e. $f \in l_1$) and where $f(z)$ is schlicht in $|z| \leqq 1$, does it follow that $f(z)$ generates $l_1$? The reader is referred to [4] for a complete explanation of this problem and all the terms involved. For our purposes it suffices to state the problem as follows: Given $\varepsilon > 0$, and $f$ satisfying the above hypotheses, is it true that

(1) There exists a polynomial $P$ such that, writing $P(f(z)) = \sum a_n z^n$, we have

$$|a_1 - 1| + \sum_{n=2}^{\infty} |a_n| < \varepsilon.$$

The previously cited paper treats in particular those $f(z)$ which map $|z| \leqq 1$ onto a Jordan domain with rectifiable boundary. Thus $f(z)$ is of bounded variation in the sense of Hardy, namely $f'(z) \in H^1$, and such $f$ are automatically $l_1$ by Hardy's inequality (see [6]). Even in this case, [4] gives a rather incomplete answer when $f'(z)$ has a nontrivial inner factor in the sense of Beurling (see [1]). It is our purpose to supply the affirmative answer in *all* cases of an $f(z)$ of bounded variation. Thus our main result is as follows:

THEOREM. *If $f(z)$ is schlicht in $|z| \leqq 1$ and of bounded variation, then $f$ generates $l_1$ (i.e. (1) holds).*

In what follows we let $I(z)$ denote the inner factor of $f'(z)$ (we allow the case where $I(z)$ is trivial, i.e. a constant). We also denote $\| \sum b_n z^n \| = \sum |b_n|/(n+1)$ and we write $A$ to represent a positive constant, not always the same. ($A$ may depend on $f(z)$ but is otherwise absolute.)

LEMMA 1. $|I(z)| > A(1 - |z|)^2$ *for* $|z| < 1$.

**Proof.** This is proved in [4] but we repeat the proof for the sake of completeness. $f(z)$ is schlicht, and so, by the distortion theorem $|f'(z)| > A(1 - |z|)$, also

Received by the editors September 24, 1962.

(1) This paper is an addendum to a previous paper published in the Trans. Amer. Math. Soc. **107** (1964), 466–484 by D. Newman, J. T. Schwartz and H. S. Shapiro entitled ***On generators of the Banach algebras $l_1$ and $L_1(0, \infty)$.***

$$\frac{f'(z)}{I(z)} \in H^1$$

and so

$$\left|\frac{f'(z)}{I(z)}\right| < \frac{A}{1-|z|}.$$

The result now follows by division of these two inequalities.

LEMMA 2.

$$\|F(z)\| \leqq A\left(\operatorname*{Max}_{|z|\leqq r}|F(z)| + \sqrt{(1-r)}\operatorname*{Sup}_{|z|<1}|F(z)|\right)$$

*for any* $r$, $0 < r < 1$.

**Proof.** The estimate

$$\|F(z)\| \leqq A \operatorname*{Sup}_{|z|<1}|F(z)| \tag{2}$$

is an obvious one, resulting e.g. from Schwarz' inequality. Now, writing $F(z) = \sum C_n z^n$, we have

$$\sum_{n\leqq 1/(1-r)} \frac{|C_n|}{n+1} \leqq A \sum_{n\leqq 1/(1-r)} \frac{|C_n|}{n+1} r^n \leqq A\,\|F(rz)\|$$

so that, applying (2), we obtain

$$\sum_{n\leqq 1/(1-r)} \frac{|C_n|}{n+1} \leqq A \operatorname*{Max}_{|z|\leqq r}|F(z)|. \tag{3}$$

Next the Schwarz inequality gives

$$\sum_{n>1/(1-r)} \frac{|C_n|}{n+1} \leqq \sqrt{\left(\sum_{n>1/(1-r)}|C_n|^2 \cdot \sum_{n>1/(1-r)}\frac{1}{(n+1)^2}\right)}$$
$$\leqq A\sqrt{(1-r)}\sqrt{(\textstyle\sum|C_n|^2)}$$

and since $\sqrt{(\sum|C_n|^2)} \leqq \operatorname{Sup}_{|z|<1}|F(z)|$ we obtain

$$\sum_{n>1/(1-r)} \frac{|C_n|}{n+1} \leqq A\sqrt{(1-r)} \operatorname*{Sup}_{|z|<1}|F(z)| \tag{4}$$

adding (2) and (4) now yields the result.

COROLLARY. *If* $(1-|z|)\,|F(z)| \leqq \delta$ *and* $|F(z)| \leqq M$ *in* $|z|<1$ *then* $\|F(z)\| \leqq AM^{2/3}\delta^{1/3}$.

**Proof.** Choose $r = 1-(\delta/M)^{2/3}$ in Lemma 2.

LEMMA 3. *Let* $0<\delta<1$, $\rho_i = 1-\delta^{4^i}$, *for* $i = 1, 2, \cdots, 8$, $g(z) = [I(\rho_1 z)I(\rho_2 z)\cdots I(\rho_8 z)]^{-1/8}$, *then* $\|g(z)I(z)-1\| < A\sqrt[3]{\delta}$.

**Proof.** For convenience we write $U(z)=I^{1/8}(z)$. Then $g(z)I(z)-1=\sum_{i=1}^{8}F_i(z)$, where

$$F_i(z)=[U(\rho_1 z)\cdots U(\rho_i z)]^{-1}U^i(z)-[U(\rho_1 z)\cdots U(\rho_{i-1}z)]^{-1}U^{i-1}(z).$$

We will now estimate each of the $\| F_i(z)\|$ by our corollary to Lemma 2. To do so notice that

$$|F_i(z)| \leqq |U(\rho_1 z)\cdots U(\rho_i z)|^{-1}\,|U(z)-U(\rho_i z)| \tag{5}$$

and since, by Lemma 1, $|U(\rho_i z)| \geqq A(1-\rho_i)^{1/4}$, we obtain

$$|F_i(z)| \leqq A[(1-\rho_1)\cdots(1-\rho_i)]^{-1/4}\,|U(z)-U(\rho_i z)|. \tag{6}$$

In particular, then, since $|U(z)| \leqq 1$, we have

$$|F_i(z)| \leqq A[(1-\rho_1)\cdots(1-\rho_i)]^{-1/4}. \tag{7}$$

Also, since $|U(z)| \leqq 1$, it follows that $|U'(z)| \leqq 1/(1-|z|)$ (see [2]), and so

$$|U(z)-U(\rho_i z)| = \left|\int_{\rho_i z}^{z} U'(\zeta)\,d\zeta\right| \leqq \frac{1-\rho_i}{1-|z|}. \tag{8}$$

Combining (6) and (8) gives

$$(1-|z|)\,|F_i(z)| \leqq A[(1-\rho_1)\cdots(1-\rho_{i-1})]^{-1/4}(1-\rho_i)^{3/4}. \tag{9}$$

Estimates (7) and (9) now allow us to apply the corollary to Lemma 2. The conclusion is

$$\| F_i(z)\| \leqq A[(1-\rho_1)\cdots(1-\rho_{i-1})]^{-1/4}(1-\rho_i)^{1/12} = A\sqrt[3]{\delta}. \tag{10}$$

Since (10) holds for every $i=1,2,\cdots,8$, our lemma follows.

**Proof of Theorem.** The remainder of the proof now follows the lines set down in [4], but again we will present it here for the sake of completeness.

Let $g(z)$ be the function given by Lemma 4 (clearly $g\in H^1$). Since $f'(z)/I(z)$ is outer and $H^1$ there exists a polynomial $R(z)$ such that $R(f'/I)-g$ is small in the $H^1$ norm (see [3]). Thus

$$\int \left| \frac{Rf'}{I-g} \right| < \delta$$

or, equivalently,

$$\int |Rf'-gI| < \delta. \tag{11}$$

By the theorem of Carathéodory-Walsh (see [5]), however, there is a polynomial in $f$, which we can denote as $P'(f)$, such that, uniformly,

$$|P'(f)-R| < \delta. \tag{12}$$

Thus, since $f' \in H^1$,

$$\int |P'(f)f' - Rf'| < A\delta. \tag{13}$$

Combining (11) and (13) gives

$$\int |P'(f)f' - gI| < A\delta. \tag{14}$$

Hardy's inequality (see [6]), however, states that $\|h\| \leqq A \int |h|$ and so we may conclude from (14) that

$$\|P'(f)f' - gI\| < A\delta. \tag{15}$$

From (15) and Lemma 3 it follows that

$$\|P'(f)f' - 1\| < A\sqrt[3]{\delta}, \tag{16}$$

or,

$$\left\|\frac{d}{dz}(P(f(z)) - z)\right\| < A\sqrt[3]{\delta} < \varepsilon \tag{17}$$

by choosing $\delta$ small enough. The statement (17), however, is exactly the statement (1) and the proof is complete.

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